

## Matrices

Basic matrix operations should feel fairly intuitive, with perhaps the exception of multiplication. Once again, I feel that the general cases are simple enough to remember and provide enough insight to give you a good understanding. As a result, examples will be kept to a minimum for the simple operations.

Ensure you are familiar with the terminology as presented in your notes before you continue, e.g., you know that an  $m \times n$  is a matrix of  $m$  rows and  $n$  columns, etc.

### **Sum of two matrices**

General case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Example of adding matrices:

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} &= \begin{bmatrix} 1+9 & 2+8 \\ 3+7 & 4+6 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} \end{aligned}$$

### **Difference of two matrices**

General case:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}$$

Example of subtracting matrices:

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} &= \begin{bmatrix} 1-9 & 2-8 \\ 3-7 & 4-6 \end{bmatrix} \\ &= \begin{bmatrix} -8 & -6 \\ -4 & -2 \end{bmatrix} \end{aligned}$$

### **Scalar Product of a matrix**

General case:

Let  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$sA = s \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ = \begin{bmatrix} sa_{11} & sa_{12} \\ sa_{21} & sa_{22} \end{bmatrix}$$

## Negative of a matrix

This is basically a special case of scalar product. That is, if the scalar used is -1, then each component will be inverted and the result will be the negative of the matrix.

$$-A = (-1)A$$

## Multiplication

Multiplication can be tricky and requires a good deal of practice to build up confidence – it can be easy to make silly errors if you're not careful.

Before you can go about performing this operation, you have to be aware that matrices have to satisfy certain conditions before they can be multiplied. Basically,  $m \times n$  and  $n \times p$  can be multiplied because the number of columns in the first matrix is equal to the number of rows in the second matrix.

The order in which matrices is also important, i.e.,  $n \times p$  and  $m \times n$  can not be multiplied in that order because  $p \neq m$ .

In my opinion, the easiest way to check is to write the sizes under the respective matrices. If the two inner values match, then the multiplication can go ahead. And if so, then the outer values give the size of the resulting matrix. E.g.,

$$\begin{array}{ccc} 4 \times 2 & & 2 \times 1 \\ & \underbrace{\hspace{2cm}} & \\ & \text{equal} & \\ & \underbrace{\hspace{2cm}} & \\ & \text{size of result} & \end{array}$$

From the example above, we can imagine we have a 4x2 and a 2x1 matrix. After explicitly writing out their sizes, we can see that the inner values do match, and so the resulting matrix will be 4x1.

General case (for 2x2 matrices):

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Admittedly, this general case is not as intuitive as the others. It may not be immediately obvious as to what is going on. If I were to describe the process in words, then to work out the result of the component in the first row and the first column, multiply the first row by the second column. For the component in the first

row and second column, then multiply the first row by the second column etc. Of course, the tricky bit is that multiplication step. To multiply a row by a column, what to do is multiply the first component in the row by the first component in the column. Then multiply the second component in the row by the second component in the column, and add to the previous product and so on until there are no more components. That may not be any clearer, so that's why plenty of practice is needed.

Examples of matrix multiplication:

$$\text{Let } A = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{a) } AB &= \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (4 \times -3) + (-1 \times 1) & (4 \times 2) + (-1 \times 2) \\ (2 \times -3) + (0 \times 1) & (2 \times 2) + (0 \times 2) \end{pmatrix} \\ &= \begin{pmatrix} -13 & 6 \\ -6 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b) } BA &= \begin{pmatrix} -3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (-3 \times 4) + (2 \times 2) & (-3 \times -1) + (2 \times 0) \\ (1 \times 4) + (2 \times 2) & (-1 \times 1) + (2 \times 0) \end{pmatrix} \\ &= \begin{pmatrix} -8 & 3 \\ 8 & -1 \end{pmatrix} \end{aligned}$$

As you can see  $AB \neq BA$ .

c) For a final example which I hope many of you can relate to a little easier, imagine we have a football league represented by a matrix, with the rows representing a team, and the columns showing their current performance (i.e., wins, draws and losses), and then we have a matrix for the points awarded for their performance:

$$\begin{array}{l} \text{Arsenal} \\ \text{ManU} \\ \text{Spurs} \\ \text{Chelsea} \end{array} \begin{array}{c} w \quad d \quad l \\ \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 4 \\ 3 & 1 & 1 \\ 1 & 4 & 0 \end{pmatrix} \end{array} \quad \text{points matrix} = \begin{array}{c} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \\ \text{win} \\ \text{draw} \\ \text{lose} \end{array}$$

Now, we can multiply these matrices to obtain the number of points each team has currently won.

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 4 \\ 3 & 1 & 1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (2 \times 3) + (2 \times 1) + (1 \times 0) \\ (0 \times 3) + (1 \times 1) + (4 \times 0) \\ (3 \times 3) + (1 \times 1) + (1 \times 0) \\ (1 \times 3) + (4 \times 1) + (0 \times 0) \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \\ 7 \end{pmatrix}$$